ON THE PROPAGATION OF ELASTO-PLASTIC WAVES FOR COMBINED STRESSES

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In his paper [1] Rakhmatulin considered several problems of the propagation of elasto-plastic waves in the presence of combined stresses. He studied the case when the wave of the combined stress is a wave of a strong discontinuity, propagating with a velocity which is smaller than the velocity of the usual elasto-plastic waves (Riemann waves). Thus, Rakhmatulin assumes that in the case of a combined stress for a shock in a plastic body there propagates at first a group of plastic waves and then the wave of the strong discontinuity which is a wave of a combined stress.

The present paper studies the same problem on the basis of the equations established by Rakhmatulin; however, it also considers another possible case of propagation which may occur for certain materials. For example, it is shown that for such materials the combined dynamic stress is transmitted in a plastic body only by combined waves. These waves propagate in a body faster than the usual plastic waves [2,3]. The presented study is rather qualitative than quantitative, since the theory of small elasto-plastic deformations, used in this investigation, has not yet been verified experimentally (and adapted in a proper manner) for dynamic problems.

Consider the problem of compression-shearing stresses of two free strips, formulated by Rakhmatulin. It will be assumed that the material of the strips is elasto-plastic and satisfies the equations of the theory of small elasto-plastic deformations. Introducing certain physically justified simplifications, Rakhmatulin reduced these equations to the form of equations (5.5) of reference [1]:

$$\begin{gathered}
\frac{2}{3} X_x - \frac{1}{3} Y_y = \frac{2}{3} \frac{\sigma_i}{e_i} \left(\frac{2}{3} e_{xx} - \frac{1}{3} e_{zz} \right) \\
-\frac{1}{3} (X_x + Y_y) = \frac{2}{3} \frac{\sigma_i}{e_i} \left(\frac{2}{3} e_{zz} - \frac{1}{3} e_{xx} \right) \\
\frac{1}{3} (X_x + Y_y) = k (e_{xx} + e_{zz}) \\
e_i = \frac{\sqrt{2}}{3} \sqrt{\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial x} - e_{zz} \right)^2 + e_{zz}^2 + \frac{3}{2} \left(\frac{\partial v}{\partial x} \right)^2}
\end{gathered} \tag{1}$$

The stress intensity σ_i is represented as a function of the strain intensity

$$\sigma_i = \sigma_i (e_i) \tag{2}$$

The equations (1) and (2) have to be supplemented by the equations of motion which may be written in the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x}, \qquad \rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial Y_x}{\partial x} \tag{3}$$

since it is assumed that the displacement components u and v depend only on the single spacecoordinate x and the time t:

$$u = u(x, t), \qquad v = v(x, t) \tag{4}$$

In all the above formulas all the components are "average" components which have been denoted by Rakhmatulin with the superscript ^{0}. For the sake of simplicity this index has been omitted here.

To facilitate the manipulations it has been assumed in what follows that the material is incompressible:

$$e_{xx} + e_{zz} = 0 \tag{5}$$

Using the condition of incompressibility (5), the system (1) may be reduced to the form

$$\begin{gathered}
2X_x - Y_y = 2 \frac{\sigma_i}{e_i} e_{xx}, \qquad X_x + Y_y = 2 \frac{\sigma_i}{e_i} e_{xx} \\
e_i = \frac{1}{\sqrt{3}} \sqrt{4 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2}
\end{gathered} \tag{6}$$

where $e_{xx} = \partial u / \partial x$ and $e_{xy} = \partial v / \partial x$ are the strain components.

The stress components may be expressed in terms of the strain components in the following manner:

$$X_x = \frac{4}{3} \frac{\sigma_i}{e_i} \frac{\partial u}{\partial x}, \qquad Y_y = \frac{2}{3} \frac{\sigma_i}{e_i} \frac{\partial u}{\partial x}, \qquad Y_x = \frac{1}{3} \frac{\sigma_i}{e_i} \frac{\partial v}{\partial x} \tag{7}$$

It will be assumed that the function in (2) is monotonically increasing and that the curve forms an angle with the Oe_i axis (in particular, it may be, for example, the exponential function).

Introducing (7) in (3), one obtains

$$\rho \frac{\partial^2 u}{\partial t^2} = L\frac{\partial^2 u}{\partial x^2} + 4N\frac{\partial^2 v}{\partial x^2}, \qquad \rho\frac{\partial^2 v}{\partial t^2} = M\frac{\partial^2 v}{\partial x^2} + N\frac{\partial^2 u}{\partial x^2} \tag{8}$$

where

$$L = \frac{4}{3}\left[\frac{\sigma_i}{e_i} + \frac{4}{3e_i^3}\left(\frac{\partial u}{\partial x}\right)^2 (e_i\sigma_i' - \sigma_i)\right], \qquad N = \frac{4}{9e_i^3}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}(e_i\sigma_i' - \sigma_i)$$
$$M = \frac{1}{3}\left[\frac{\sigma_i}{e_i} + \frac{1}{3e_i^3}\left(\frac{\partial v}{\partial x}\right)^2 (e_i\sigma_i' - \sigma_i)\right], \tag{9}$$

Consequently, by means of the formulas (2) and (7) functions L, M, N may be expressed in terms of $\partial u/\partial x$ and $\partial v/\partial x$ only.

Equations (8) are the equations of motion, corresponding to the equations (1.6) of Rakhmatulin [1]. It follows from (8) that for the problem under consideration one has in the general case two types of waves. The usual assumption will be made that for passage through any wave front the increments

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial t}dt, \qquad du_t = \frac{\partial u_t}{\partial x}dx + \frac{\partial u_t}{\partial t}dt$$
$$dv_x = \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial t}dt, \qquad dv_t = \frac{\partial v_t}{\partial x}dx + \frac{\partial v_t}{\partial t}\partial t$$

taken along the front remain continuous; equations (8) will now be supplemented by the equations

$$\frac{\partial^2 u}{\partial x^2}dx^2 - \frac{\partial^2 u}{\partial t^2}dt^2 = du_x dx - du_t dt, \qquad \frac{\partial^2 v}{\partial x^2}dx^2 - \frac{d^2 v}{dt^2}dt^2 = dv_x dx - dv_t dt \tag{10}$$

Solving the system, consisting of equations (8) and (10), for the highest order derivatives, one readily finds the characteristic equation

$$\rho^2\left(\frac{dx}{dt}\right)^4 - \rho(L+M)\left(\frac{dx}{dt}\right)^2 + LM - 4N^2 = 0 \tag{11}$$

and the differential expressions satisfied by the characteristics. There are four such expressions, but, using (11), only one of them is found to be independent, for example

$$(cdu_x - du_t)N + (\rho c^2 - L)(cdv_x - dv_t) = 0 \tag{12}$$

Here c is the velocity of propagation of the plastic wave front at which the expression (12) is satisfied. As follows from (11), the velocity may have two values

$$\left.\begin{aligned} c_I^2 &= c_I^2\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) \\ c_{II}^2 &= c_{II}^2\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)\end{aligned}\right\} = \left(\frac{dx}{dt}\right)^2 = \frac{L + M \pm \sqrt{(L - M)^2 + 16N^2}}{2\rho} \tag{13}$$

Since, obviously $L > 0$ and $M > 0$, it follows that

$$c_I > c_{II} \tag{14}$$

The velocities determined by the relations (13) are the velocities of propagation of the two types of waves of combined stress. These types of waves will be denoted by (I) and (II), respectively. In particular, in regions where one may assume $v = 0$, the waves will propagate with velocity

$$c_{I1}{}^2 = \frac{4}{3}\frac{\sigma_{i1}{}'}{\rho} \tag{15}$$

of Riemann waves (I1); here (σ'_{i1}) is calculated from the formally stated expression (2)

$$\sigma_{i1} = \sigma_{i1}(e_{i1})$$

however, in actual fact, these expressions are different.

Finally, in regions where one may assume $u = 0$, only Riemann waves (II 2) will propagate with velocity

$$c_{II2}{}^2 = \frac{1}{3}\frac{\sigma_{i2}{}'}{\rho} \tag{16}$$

Here (σ'_{i2}) is calculated from the other expression of the type $\sigma_{i2} = \sigma_{i2}(e_{i2})$.

It will be shown that for certain conditions, i.e. for certain mechanical properties of the plastic material

$$c_I^2 > \frac{L}{\rho} > c_{I1}{}^2 \tag{17}$$

The first part of the inequality (17) follows directly from (13); for the proof of the second part one must compare the expression L of the relations (9) with $c_{I_1}{}^2$ of (15), i.e. one must have

$$\frac{1}{3}\left[4\,\frac{\sigma_i}{e_i} + \frac{16}{3e_i^3}\left(\frac{\partial u}{\partial x}\right)^2 (e_i\sigma_i' - \sigma_i)\right] > \frac{4}{3}\sigma_{i1}'$$

or

$$\frac{1}{3e_i^2}\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\sigma_i}{e_i} - \sigma_i'\right) + \sigma_i' > \sigma_{i1}' \tag{18}$$

The inequality (18) is satisfied for very many types of materials for which as a rule $\sigma_i/e_i \gg \sigma_i'$ and σ_i' does not differ greatly from σ'_{i1}.

One may draw from (17) the important conclusion that in materials, satisfying the former conditions, waves of combined stress (I) propagate faster than the standard plastic Riemann waves (I1). In fact, if one compares the velocity c_I with the velocity c_{I1} of the elastic waves, one obtains $c_I < c_{I\,1el}$, but in certain cases these two velocities are almost equal. Consequently, for instantaneous combined stress at the end of a plastic body one must assume that the wave I is a wave of a strong discontinuity.

This reasoning does not apply to the velocity c_{II}, because it follows readily from (13) that $c_{II}^2 < M/\rho$, and comparing M/ρ with $c_{II\,2}^2$, one obtains $c_{II\,2}^2 < M/\rho$ and, consequently, in this way one does not obtain comparison of c_{II} with $c_{II\,2}$. Comparing c_{II} directly with $c_{II\,2}$ and making the same assumptions as above, one may show that

$$c_{II_2}{}^2 < c_{II}{}^2 \tag{19}$$

and therefore, the wave (II 2) propagates more slowly than the wave II.

It will now be shown that the waves I and II are actually waves of combined stress and not the usual plastic waves. Let α and β denote the discontinuities of the derivatives $\partial^2 u/\partial x^2$ and $\partial^2 v/\partial x^2$ for the passage through the front of the wave

$$\alpha = \left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 u}{\partial x^2}\bigg|_+ - \frac{\partial^2 u}{\partial x^2}\bigg|_-, \qquad \beta = \left[\frac{\partial^2 v}{\partial x^2}\right] = \frac{\partial^2 v}{\partial x^2}\bigg|_+ - \frac{\partial^2 v}{\partial x^2}\bigg|_-$$

If this front is a front of the wave (I), then these discontinuities will be denoted by α_I and β_I, while if it is of the type II, they will be denoted by α_{II} and β_{II}. It follows from (10) that between the discontinuities of the second order derivatives there is a relationship

$$\left[\frac{\partial^2 u}{\partial x^2}\right] c^2 - \left[\frac{\partial^2 u}{\partial t^2}\right] = 0, \qquad \left[\frac{\partial^2 v}{\partial x^2}\right] c^2 - \left[\frac{\partial^2 v}{\partial t^2}\right] = 0 \tag{20}$$

and from (8), taking into consideration (20), one obtains the expressions

$$(\rho c^2 - L)\,\alpha - 4N\beta = 0, \qquad N\alpha - (\rho c^2 - M)\,\beta = 0 \tag{21}$$

Equations (21) are not independent, since it follows from (11) that

$$\frac{\rho c^2 - L}{N} = \frac{4N}{\rho c^2 - M} \tag{22}$$

Expressions (21) and (12) are likewise not independent, if one takes (10) into consideration. Consequently, at the front of the wave (I), one has

$$(\rho c_I{}^2 - L)\,\alpha_I - 4N\beta_I = 0 \tag{23}$$

and at the front of the wave (II)

$$(\rho c_{II}^2 - L)\,\alpha_{II} - 4N\beta_{II} = 0 \tag{24}$$

It follows from (23) and (24) that at the front of a wave I, as well as at the front of a wave II, all second order derivatives of u and v are continuous. Consequently, both waves are waves of combined stress. These waves of combined stress degenerate into the usual plastic waves only for $N = 0$. This may occur in two cases.

In the first case, one of the displacement components is zero. Hence $\partial u/\partial x = 0$ or $\partial v/\partial x = 0$. Then the system (8) reduces to a single equation: the second or the first of the equations (8) respectively (where $N = 0$). The differential relation (12), satisfied by the characteristics, reduces to one of the known relations

$$dv_t = c_{II2} dv_x \qquad \text{or} \;\; du_t = c_{I1}\, du_x$$

Consequently, in this case, in a plastic body only one type of the ordinary waves propagate; the velocity of propagation is determined by (16) or (15) respectively.

The second case, when $N = 0$, is the case of the elastic body ($\sigma_i/e_i = \sigma'_i$). In this case, the system (8) reduces to the two traditional equations of the propagation of two types of elastic waves and the velocities of propagation reduce to the known constant velocities.

The discontinuities at these two fronts of the usual waves are independent in the sense that they are not interconnected by any relations (for example, of the type of the relations (12)), and propagate with different velocities.

It follows from the above statements that the combined stress propagates in a body by two groups of ordinary waves, if the body remains elastic, i.e. for a shock which does not exceed the elastic limit. For transition of the elastic limit, plastic strain propagates in the body by two types of waves of combined stress, the velocities of propagation of which are determined by the relations (13). Generally speaking, elastic waves also may propagate in front of these waves. In each case, for instantaneous combined stress in a body, there do not arise the ordinary plastic waves, since the waves of combined stress, arising simultaneously at the end of the strips, propagate faster.

If the impact stress is not a combined stress in the sense that different strain components do not propagate simultaneously and it arises gradually, and if the loading is not instantaneous, then there will propagate in the plastic body the two types of ordinary plastic waves. Depending on the boundary conditions there exists the possibility of a simultaneous spreading of the two types of ordinary plastic waves in definite regions of the plastic body. Nevertheless, in this case there will appear plastic waves of combined stress because the discontinuities of the derivative $\partial^2 u/\partial x^2$ propagate with a velocity which differs from the velocity of propagation of the derivative $\partial^2 v/\partial x^2$ and between these discontinuities there exists no connection. Thus, these waves propagate independently, although the wave which propagates faster makes the body inhomogeneous and in this sense influences the succeeding wave.

The discontinuity will now be evaluated, assuming that in the plastic body there propagate waves I as well as waves II. It will be postulated that at a certain cross-section x_0 at time t_0 there occur simultaneously two fronts of waves I and II. It will be noted that the discontinuities at the two waves fronts are not independent, since if one takes into consideration the relation $\rho(c_I^2 + c_{II}^2) = L + M$, then (23) and (24) may be written in the form

$$\frac{2N}{\rho c_I^2 - L} = \frac{\alpha_I}{2\beta_I} = -\frac{2\beta_{II}}{\alpha_{II}} \tag{25}$$

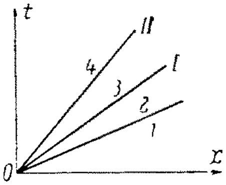

Fig. 1.

where N, L and c_I are evaluated for $x = x_0$ and $t = t_0$. The resultant discontinuity α^* of the derivative $\partial^2 u/\partial x^2$ for the passage of the two wave fronts is the sum of two discontinuities $\alpha^* = \alpha_I + \alpha_{II}$. Analogously, one obtains for the resultant discontinuity β^* of the derivative $\partial^2 v/\partial x^2$

$$\beta^* = \beta_I + \beta_{II}.$$

The coefficients of the discontinuities always satisfy the relation

$$\alpha_I \alpha_{II} - 4\beta_I \beta_{II} = 0 \tag{26}$$

If at a certain section x_0 (in particular, $x_0 = 0$) at time t_0 one knows the resultant discontinuities α^* abd β^*, and likewise the state of stress, then one finds from (25) the discontinuity for each wave separately

$$\alpha_I = \frac{\alpha^*}{2} + \beta^* \omega, \qquad \beta_I = \frac{1}{2}\left(\frac{\alpha^*}{2\omega} + \beta^*\right), \qquad \alpha_{II} = \frac{\alpha^*}{2} - \beta^* \omega \tag{27}$$

$$\beta_{II} = \frac{1}{2}\left(\beta^* - \frac{\alpha^*}{2\omega}\right) \qquad \omega = \frac{2N}{\rho c_I^2 - L}$$

It must be noted that not all discontinuities have the same sign.

For an approximate solution of the problem under consideration one may propose many methods.

If the loading is instantaneous, then it may be assumed that the waves I and II are waves of strong discontinuity, propagating with constant velocities (13). Then in the xOt plane (Fig. 1) one will have four regions: the region *1* is not deformed, the region *2* is elastically deformed and the regions *3* and *4* are plastically deformed after the passage of the waves of combined stress I and II respectively.

In region *2* the solution is known from the solution of the elastic problem. Consequently, in this region, the values of the strains are known, as well as the velocities u_{x2}, u_{t2}, v_{x2}, v_{t2}. The determination of the solution in region *3* may be carried out with the help of the expressions

$$\begin{aligned} v_{t3} + c_I v_{x3} &= v_{t2} + c_I v_{x2}, \qquad c_I \rho (v_{t3} - v_{t2}) = -X_{y3} + X_{y2} \\ u_{t3} + c_I u_{x3} &= u_{t2} + c_I u_{x2}, \qquad c_I \rho (u_{t3} - u_{t2}) = -X_{x3} + X_{x2} \end{aligned} \qquad (28)$$

to which the first expression (13) for the determination of c_I and likewise the expression (7) must be added.

The transition from the region *3* to region *4* may be effected in an analogous manner.

If the load at $x = 0$ is considered to be the consequence of instantaneous stresses, then one may act in exactly the same manner, but the xt-plane must be divided into a larger number of regions.

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